~~Resit~~ Exam — Analysis (WBMA012-05)

Thursday 23 January 2025, 18.15h–20.15h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. Provide clear arguments for all your answers: only answering "yes", "no", or "42" is not sufficient. You may use all theorems and statements in the book, but you should clearly indicate which of them you are using.
3. The total score for all questions equals 90. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (6 + 9 = 15 points)

(a) Prove that every nonempty set that has a lower bound also has a ~~least~~ greatest lower bound.

(b) Assume that $A \subset \mathbb{R}$ is nonempty and bounded from above. Show that the set

$$B = \{b \in \mathbb{R} \mid b \text{ is an upper bound for } A\}$$

is bounded below and $\sup A = \inf B$.

Problem 2 (5 + 10 = 15 points)

(a) Define what is a Cauchy sequence and state Cauchy criterion for sequences.

(b) Let $b \in \mathbb{N}$, $b \geq 2$. Consider a series of the following form

$$\sum_{\ell=-k}^{\infty} a_\ell b^{-\ell},$$

where $k \in \mathbb{N} \cup \{0\}$ and $a_\ell \in \{0, 1, \ldots, b-1\}$ for all $\ell \in \mathbb{N}$.

Show that the series converges to a real number. Note that this series is said to converge if the sequence of its partial sums (s_n), $s_n = \sum_{\ell=-k}^{n} a_\ell b^{-\ell}$, converges.

Problem 3 (10 + 5 = 15 points)

Let $A, B \subseteq \mathbb{R}$ two arbitrary sets.

(a) Prove that $\overline{A} \cup \overline{B} = \overline{A \cup B}$.

(b) Does this result extend to infinite unions of sets? That is, is it true that $\overline{\bigcup_\alpha A_\alpha} = \bigcup_\alpha \overline{A_\alpha}$? Provide a proof or a counterexample.

Please turn over for problems 4, 5 and 6!

Problem 4 (5 + 5 + 5 = 15 points)

Let $f : [1, 5] \to \mathbb{R}$ be differentiable and assume that

$$f(1) = 3, \quad f(3) = 7, \quad f(5) = 3.$$

Prove the following statements:

(a) There exists a point $a \in [1, 5]$ such that $f(a) = a$.

(b) There exists a point $b \in [1, 5]$ such that $f'(b) = 0$.

(c) There exists a point $c \in [1, 5]$ such that $f'(c) = -\frac{1}{4}$.

Problem 5 (3 + 8 + 4 = 15 points)

Le the functions $f_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, be defined by

$$f_n(x) := \frac{nx}{1 + |nx|}.$$

(a) Show that all the functions f_n are continuous.

(b) Let $f(x) := \lim f_n(x)$. For which $x \in \mathbb{R}$ is f defined? For which $x \in \mathbb{R}$ is f continuous?

(c) Is the convergence of the sequence (f_n) uniform on $\mathbb{R}$?

Problem 6 (3 + 6 + 3 + 3 = 15 points)

Let the functions $f, g : [0, \pi/2] \to \mathbb{R}$ be defined by

$$f(x) = \cos^2(x) \quad \text{and} \quad g(x) = \begin{cases} \cos^2(x) & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}.$$

Prove the following statements:

1. $L(g, P) = 0$
2. $U(g, P) = U(f, P)$
3. $U(f, P) \geq \pi/4$
4. The function g is not integrable on $[0, \pi/2]$.

Useful identity: $2\cos^2(x) = 1 + \cos(2x)$

Please do not forget to fill out the online course evaluation!

End of test (90 points)

Note that all the problems could be solved in multiple ways, and not all of those solutions are included here.

Solution of problem 1 (6 + 9 = 15 points)

(a) This is not the Axiom of Completeness, since that states that every nonempty set that has an upper bound also has a least upper bound.
Let $S \subset \mathbb{R}$ be non empty and bounded below, and consider the set $\tilde{S} = \{-s \mid s \in S\}$. By definition there is $m \in \mathbb{R}$ such that $m \leq s$ for all $s \in S$. Then $-m \geq -s$ for all $s \in S$, and thus $-m \geq \tilde{s}$ for all $\tilde{s} \in \tilde{S}$. That is, $\tilde{S}$ is bounded above.

By the Axiom of Completeness, $\tilde{S}$ has a least upper bound: $M = \sup \tilde{S}$. By the Characterization of the Least Upper Bound, $s \leq M$ for all $s \in S$ and for every $\epsilon > 0$ there is $z \in \tilde{S}$ such that $M - \epsilon < z \leq M$.
That is, $-s \geq -M$ for every $s \in S$ and for every $\epsilon > 0$ there is $s = -z \in S$ such that $-z > -M + \epsilon$, which is the Characterization of the Greatest Lower Bound for S: $-M = \inf S$.

(b) Any $b \in B$ satisfies $a \leq b$ for all $a \in A$. Therefore every $a \in A$ is a lower bound for B. This in particular shows that B is bounded below. Moreover, since A is bounded above, B is non empty as well. By the previous point, $\inf B$ exists.

Since $\inf B$ is the greatest lower bound for B, and every $a \in A$ is a lower bound for B, we have that $a \leq \inf B$ for all $a \in A$. By the Axiom of Choice we know that $\sup(A)$ exists and since every $b \in B$ is an upper bound for A, $\sup A \leq \inf B$.

Let M be an arbitrary upper bound for A. Then, by definition of the set B, $M \in B$. But then, $M \geq \inf B$. Since M was arbitrary, this shows that $\sup A \geq \inf B$, and therefore $\sup A = \inf B$.

Solution of problem 2 (6 + 9 = 15 points)

(a) A sequence (a_n) is called a Cauchy sequence if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ for all $n, m \in \mathbb{N}$ with $n, m \geq N$.

Cauchy Criterion for sequences states that a sequence (a_n) converges if and only if it is a Cauchy sequence.

(b) Let $\epsilon > 0$. Let $s_n = \sum_{\ell=-k}^{\infty} a_\ell$ be the partial sums of the series. We want to show that s_n is a Cauchy sequence and, thus, by Cauchy Criterion is convergent.

Method 1.
Since $b > 2$, the sequence $b^{-n} \to 0$ and thus, by definition of convergence, there exists $N \in \mathbb{N}$ such that $b^{-N} < \epsilon$.

For any $n > m \geq N$, we have

$$\begin{aligned}
|s_n - s_m| = \left| \sum_{j=m+1}^{n} a_j b^{-j} \right| &\leq \sum_{j=m+1}^{n} a_j b^{-j} \leq \sum_{j=m+1}^{n} (b-1) b^{-j} \\
&\leq (b-1) b^{-m-1} \sum_{j=0}^{n-m-1} b^{-j} \\
&< (b-1) b^{-m-1} \sum_{j=0}^{\infty} b^{-j} = (b-1) b^{-m-1} \frac{1}{1 - b^{-1}} = b^{-m} \\
&< b^{-N} < \epsilon,
\end{aligned}$$

and thus is a Cauchy sequence.

Method 2.
Since $b > 2$, the sequence $b^{-n} \to 0$ and thus, by definition of convergence, there exists $N \in \mathbb{N}$ such that $b^{-N} < \epsilon/(1 - b^{-1})$.

For any $n > m \geq N$, we have

$$\begin{aligned}
|s_n - s_m| = \left| \sum_{j=m+1}^{n} a_j b^{-j} \right| &\leq \sum_{j=m+1}^{n} a_j b^{-j} \leq \sum_{j=m+1}^{n} (b-1) b^{-j} \\
&\leq (b-1) b^{-m-1} \sum_{j=0}^{n-m-1} b^{-j} \\
&< (b-1) b^{-m-1} \frac{b(1 - b^{-n+m})}{b-1} = b^{-m}(1 - b^{m-n}) \\
&< b^{-N}(1 - b^{-1}) < \epsilon.
\end{aligned}$$

and thus is a Cauchy sequence.

Solution of problem 3 (10 + 5 = 15 points)

(a) Recall that the closure of a set S is the set $\overline{S} = S \cup \{\text{limit points of } S\}$.

Since $A \subseteq A \cup B$, any limit point of A is by definition a limit point of $A \cup B$. Indeed, if $V_(x)$ intersects A in a point other than x, then it also intersects $A \cup B$ in a point other than x itself. Thus $\overline{A} \subseteq \overline{A \cup B}$.

Similarly, since $B \subseteq A \cup B$, any limit point of B is by definition a limit point of $A \cup B$. Thus $\overline{B} \subseteq \overline{A \cup B}$.

Therefore $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

Since $A \cup B \subseteq \overline{A} \cup \overline{B}$, the same reasoning also implies that $\overline{A \cup B} \subseteq \overline{\overline{A} \cup \overline{B}}$.

In class we have proven that the union of a finite collection of closed sets is closed, thus in particular $\overline{A} \cup \overline{B}$ is closed. This means that $\overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$.

Wrapping it all up, we have shown that

$$\overline{A \cup B} \subseteq \overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}.$$

In summary, we have shown that $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$ and $\overline{A} \cup \overline{B} \supseteq \overline{A \cup B}$, and thus $\overline{A} \cup \overline{B} = \overline{A \cup B}$.

(b) We have seen a counterexample in class.

Let $A_n = \{1/n\}$ be the set containing the single element $1/n$ for $n \in \mathbb{N}$.

As seen in class, the set A_n, containing only the one isolated point, is closed. Alternatively, one can show this arguing that union of open sets are open and thus $\{1/n\} = ((-\infty, 1/n) \cup (1/n, \infty))^c$ is the complement of the open set $(-\infty, 1/n) \cup (1/n, \infty)$.

Thus $\bigcup_{n=1}^{\infty} \overline{A_n} = \bigcup_{n=1}^{\infty} \overline{A_n} = \{1/n \mid n \in \mathbb{N}\}$.

However, $\overline{\bigcup_{n=1}^{\infty} A_n} = \overline{\{1/n \mid n \in \mathbb{N}\}} = \{1/n \mid n \in \mathbb{N}\} \cup \{0\}$. As before one can argue by saying that we saw this in class or show it directly with the Archimedean property or other means (as we did in e.g. lectures 6 and 7).

Therefore

$$\overline{\bigcup_{n=1}^{\infty} A_n} = \{1/n \mid n \in \mathbb{N}\} \cup \{0\} \neq \{1/n \mid n \in \mathbb{N}\} = \bigcup_{n=1}^{\infty} \overline{A_n}.$$

Solution of problem 4 (5 + 5 + 5 = 15 points)

(a) We proceed as in lecture 9. Introduce the new function $g(x) = f(x) - x$. This is a continuous function due to the Algebraic Continuity Theorem. We have that $g(1) = 2 > 0$ and $g(5) = -2 < 0$, thus by the Intermediate Value Theorem there exists $a \in (1, 5)$ such that $g(a) = 0$, that is, $f(a) = a$.

(b) Since $f(1) = f(5) = 3$, Rolle's Theorem implies that there exists $b \in (1, 5)$ such that $f'(b) = 0$.

(c) By the Mean Value Theorem there exists $d \in (3, 5)$ such that

$$f'(d) = \frac{f(5) - f(3)}{5 - 3} = \frac{3 - 7}{2} = -2.$$

By the previous point we know that there exists b such that $f'(b) = 0$. Since $-2 < -1/4 < 0$, Darboux Theorem implies that there exists $c \in (1, 5)$ such that $f'(c) = -1/4$.

Solution of problem 5 (3 + 8 + 4 = 15 points)

(a) For $n \in \mathbb{N}$, the fact that the function $f_n = \frac{x}{1/n+|x|}$ is continuous follows from the Algebraic Continuity Theorem observing that x is continuous and $|x| + 1/n$ is continuous and strictly positive at all points in $\mathbb{R}$.

(b) For $x \neq 0$, by the Algebraic Limit Theorem

$$f(x) = \lim_{n\to\infty} f_n(x) = \frac{x}{\lim(|x| + 1/n)} = \frac{x}{|x| + \lim 1/n} = \frac{x}{|x|}.$$

If $x = 0$, $f(0) = \lim_{n\to\infty} f_n(0) = 0$ is well defined, thus $f(x)$ is defined for all $x \in \mathbb{R}$.

Let (a_k) and (b_k), $a_k = 1/k$ and $b_k = -1/k$, be two sequences converging to 0. Then $f(a_k) \to 1$ and $f(b_n) \to -1$, contradicting the sequential characterization of continuity. This shows that f is not continuous at 0.

Let $x, c > 0$, then $|f(x) - f(c)| = |1 - 1| = 0$, and similarly for $x, c < 0$.

In other words, for any $c \neq 0$ and $\epsilon > 0$, let $\delta = |c|$. We have shown that for all $x \in \mathbb{R}$ such that $|x - c| < \delta$ it holds $|f(x) - f(c)| = 0 < \epsilon$. That is, f is continuous at all points $c \neq 0$.

(c) *Method 1.*
The preservation of continuity states that uniform convergence preserves continuity. That is, if assume $f_n : A \to \mathbb{R}$ is such that $f_n \to f$ uniformly on A, and each f_n is continuous at $c \in A$, then f is continuous at c.

In the first point of the exercise we have observed that the f_n are continuous but f is not continuous at 0, thus f_n cannot converge uniformly to f.

Method 2.
If the convergence is uniform then for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$ and all $x \in \mathbb{R}$.

Let $\epsilon = 1/2$. Then for any n, taking $x = 0$ we have $|f_n(0) - f(0)| = 1 > \epsilon$. Therefore the convergence is not uniform.

Method 3.
Note that for all $n \in \mathbb{N}$

$$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup_{x\in\mathbb{R}} \left| \frac{nx}{1 + n|x|} - \frac{x}{|x|} \right| = \sup_{x\in\mathbb{R}} \left| \frac{-x}{|x|(1 + n|x|)} \right| = \sup_{x\in\mathbb{R}} \frac{1}{1 + n|x|} = 1.$$

Therefore

$$\lim_{n\to\infty} \left(\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| \right) = 1 \neq 0.$$

That is, the convergence is not uniform.

Solution of problem 6 (3 + 6 + 3 + 3 = 15 points)

(a) For any partition $P = \{0 = x_0 < x_1 < \ldots < x_n = \pi/2\}$ of the interval $[0, \pi/2]$, since any subinterval $[x_{k-1}, x_k]$ contains at least one irrational point (at which g is zero), we have

$$m_k = \inf\{g(x) \mid x \in [x_{k-1}, x_k]\} = 0.$$

That is

$$L(g, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = 0.$$

(b) For any subinterval $[x_{k-1}, x_k]$, denote

$$M_k = \sup\{g(x) \mid x \in [x_{k-1}, x_k]\} \quad \text{and} \quad M'_k = \sup\{f(x) \mid x \in [x_{k-1}, x_k]\}.$$

Since f is decreasing in the interval $[0, \pi/2]$, we have that $M'_k = f(x_{k-1})$ for all $k = 1, \ldots, n$.
(1 point)

If (a_n) is a sequence of rational numbers in the subinterval converging to x_{k-1}, then

$$\lim g(a_n) = \lim \cos^2(a_n) = \cos^2(x_{k-1}) = M'_k$$

for all $n \in \mathbb{N}$, where the second equality follows from the Algebraic Continuity Theorem and the fact that the cosine is a continuous function.

On the other hand, by definition $g(x) \leq f(x) \leq M'_k$ for all $x \in [x_{k-1}, x_k]$, and thus $M_k = M'_k$.

This shows that

$$U(g, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = \sum_{k=1}^{n} M'_k(x_k - x_{k-1}) = U(f, P).$$

(c) Since (as argued above) f is continuous on $[0, \pi/2]$, it is integrable on the same interval. Therefore we have that

$$\int_0^{\pi/2} f(x)\, dx \leq U(f, P).$$

By the Fundamental Theorem of Calculus we have

$$\int_0^{\pi/2} f(x)\, dx = \int_0^{\pi/2} cos^2(x)\, dx = \frac{1}{2}\int_0^{\pi/2} (1+\cos(2x))\, dx = \frac{1}{2}\left[x + \frac{1}{2}\sin(2x)\right]_0^{\pi/2} = \frac{\pi}{4}.$$

(d) For any partition P of $[0, \pi/2]$ we have shown that

$$U(g, P) - L(g, P) = U(f, P) \geq \frac{\pi}{4}.$$

Therefore, the function g is not integrable on $[0, \pi/2]$.